

Better Actions*

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We explain why compact $U(1)$ confines and how to fix it. We show that plaquettes of negative trace carry most of the confinement signal in compact $SU(2)$. We show how to perform noncompact gauge-invariant simulations without auxiliary fields. We suggest a way to simulate fermions without doublers.

1. INTRODUCTION

We reort here the results of four different studies all of which are attempts to find better lattice actions. The first study explains why compact $U(1)$ gauge theory displays confinement at strong coupling and shows how to remove this artifact. The second study shows that most of the confinement signal in compact $SU(2)$ gauge theory is due to plaquettes of negative trace. These results obtain for both the Wilson action and the Manton action. The third study shows how to perform noncompact lattice simulations that are exactly gauge invariant and that do not involve auxiliary fields. The fourth study presents a lattice action for fermions that avoids doublers.

2. WHY COMPACT $U(1)$ CONFINES

Compact lattice simulations of $U(1)$ gauge theory display confinement at strong coupling, as shown by Figure 1 which plots Creutz ratios[1] obtained with the Wilson action. This lattice artifact is obvious in the figure at $\beta = 0.25$ and at $\beta = 0.5$, and incipient at $\beta = 1$. It arises because $U(1)$ is a circle; if one cuts the circle, then there is no confinement, as shown in the figure by the $\chi(i, j)$'s labeled "cut \bigcirc " which follow the curves of the exact Creutz ratios down to $\beta = 0.25$.

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We ran on a 12^4 lattice and began all runs from a cold start in which all links were unity. We used a Metropolis algorithm and rejected any plaquette whose phase θ was either greater than $\pi - \epsilon$ or less than $-\pi + \epsilon$. We saw no confinement signal as long as the step size was smaller than the thickness 2ϵ of the cut in the phase θ . Simulations with Manton's action show similar results.

One may interpret these results in terms of monopoles. Since the phase θ of each plaquette is required to lie between $\pi - \epsilon$ and $-\pi + \epsilon$, it follows that for $\epsilon > 0$, no string can ever penetrate any plaquette. For the Wilson action we took $.02 < \epsilon < 0.1$ and noticed no sensitivity to ϵ within that range. For the Manton action, we took $\epsilon = 0.026$ in all runs with cut circles.

3. COMPACT $SU(2)$

In view of these results for $U(1)$, one might wonder whether similar lattice artifacts exist in the case of the group $SU(2)$. Inasmuch as $U(1)$ and $SU(2)$ have different first homotopy groups ($\pi_1(U(1)) = Z$ but $\pi_1(SU(2)) = 0$), one might assume that excising a small cap around the antipode ($g = -1$) on the $SU(2)$ group manifold (the three sphere S_3 in four dimensions) would have little effect on Creutz ratios.

To check this assumption, we ran from cold starts on an 8^4 lattice and used a Metropolis algorithm in which we rejected plaquettes that lay within a small cap around the antipode. The step size was small compared to the size of the excluded cap. As shown in Figure 2, the Creutz ratios $\chi(i, j)$ do not depend upon whether the small

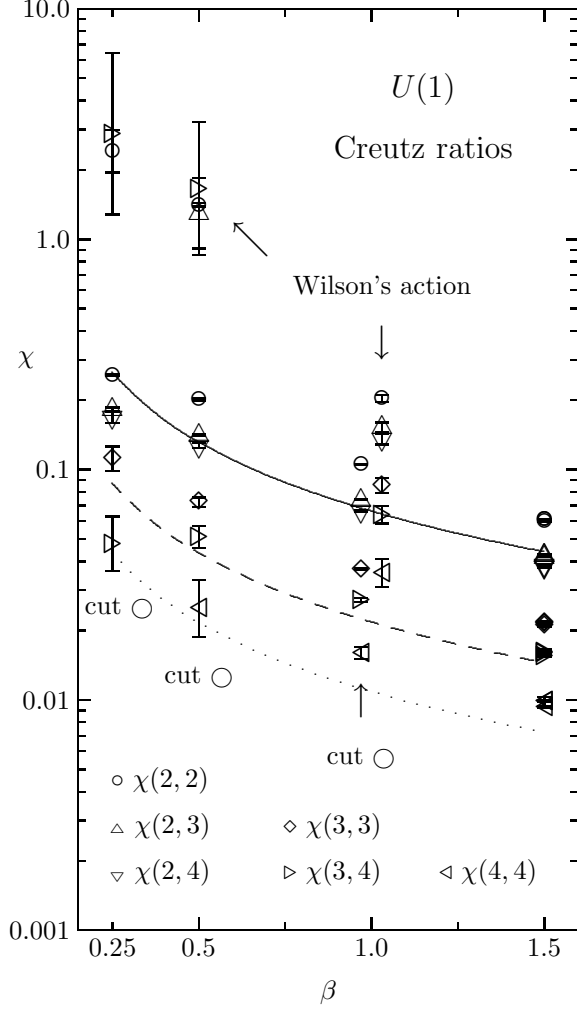


Figure 1. The $U(1)$ Creutz ratios $\chi(i, j)$ as given by Wilson's action and by Wilson's action on a cut circle. The curves represent the exact Creutz ratios, $\chi(2, 2)$ (solid), $\chi(3, 3)$ (dashes), and $\chi(4, 4)$ (dots). At $\beta = 1$, the symbols for the Wilson action are plotted to the right of those for the cut circle; at $\beta = 1.5$ they overlap. Wilson's action on the full circle confines for $\beta < 1$.

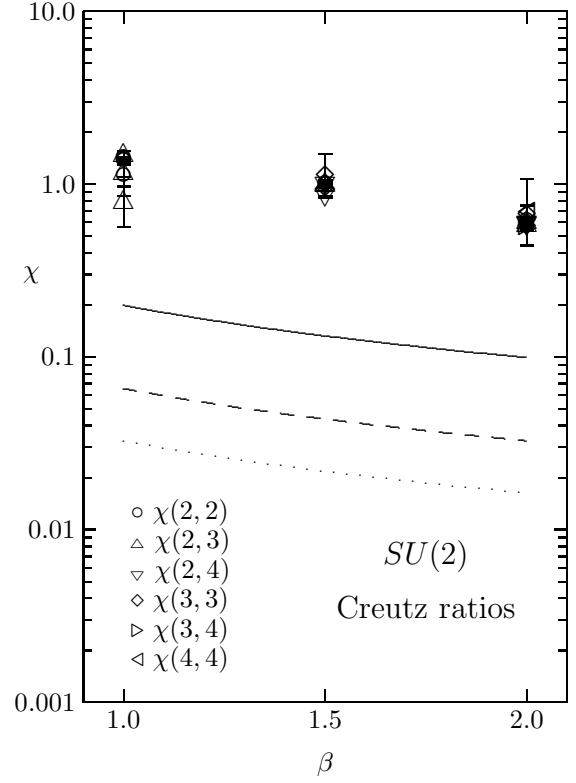


Figure 2. The $SU(2)$ Creutz ratios $\chi(i, j)$ as given by Wilson's action with and without the exclusion of a small part of the $SU(2)$ sphere. The curves represent the perturbative Creutz ratios, $\chi(2, 2)$ (solid), $\chi(3, 3)$ (dashes), and $\chi(4, 4)$ (dots). The $\chi(i, j)$'s substantially overlap.

cap was excluded.

But what about the excision of a large cap? To study this question, we again began with cold starts on an 8^4 lattice and employed a Metropolis algorithm with a small step size. We rejected all plaquettes that had a negative trace, thus excluding half of the $SU(2)$ sphere.

In Figure 3 we plot the Creutz ratios $\chi(i, j)$ both for the usual Wilson action and for the positive-plaquette Wilson action. As shown in the figure, the $\chi(i, j)$'s of the positive-plaquette simulations exhibit behavior that is substantially

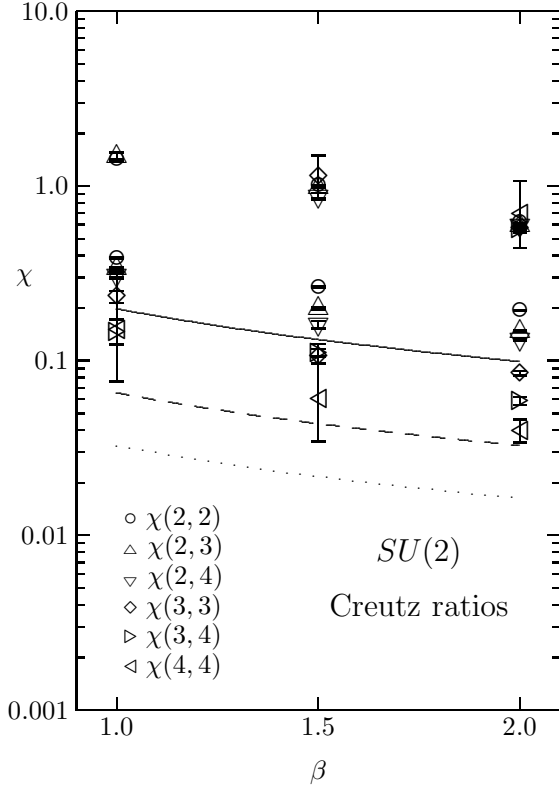


Figure 3. The $SU(2)$ Creutz ratios $\chi(i, j)$ as given by Wilson's action with and without the exclusion of half the $SU(2)$ sphere. The curves represent the perturbative Creutz ratios, $\chi(2, 2)$ (solid), $\chi(3, 3)$ (dashes), and $\chi(4, 4)$ (dots). In the two cases, the $\chi(i, j)$'s substantially differ.

perturbative. The confinement signal has disappeared. Apparently the plaquettes of negative trace carry most of the confinement signal.

4. NONCOMPACT SIMULATIONS

Suppose we write the fermion field

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \quad (1)$$

in terms of some orthonormal basis vectors $e_a(x)$ that may vary with position and time

$$\psi(x) = \psi_a(x)e_a(x). \quad (2)$$

Then the derivative $\partial_\mu \psi(x)$ has two terms:

$$\partial_\mu \psi(x) = e_a(x)\partial_\mu \psi_a(x) + \psi_a(x)\partial_\mu e_a(x). \quad (3)$$

And so if we let the gauge field $A_\mu^{ab}(x)$ be

$$A_\mu^{ab}(x) = (-i/g)e_a^\dagger(x) \cdot \partial_\mu e_b(x), \quad (4)$$

then the free action density $i\bar{\psi}\gamma^\mu \partial_\mu \psi$ becomes

$$\mathcal{L}_D = i\bar{\psi}_a\gamma^\mu (\delta_{ab}\partial_\mu + iA_\mu^{ab})\psi_b = i\bar{\psi}_a\gamma^\mu D_\mu^{ab}\psi_b \quad (5)$$

as in a gauge theory.

Under a gauge transformation

$$\psi'_a(x) = g_{ab}(x)\psi_b(x), \quad (6)$$

the field $\psi(x)$ and therefore the action is invariant if the vectors $e_a(x)$ transform as

$$e'_a(x) = g_{ca}^{-1}(x)e_c(x) \quad (7)$$

which for unitary groups is $e'_a(x) = g_{ac}^*(x)e_c(x)$.

To generate $U(1)$ gauge fields, one may use a single normalized complex three-vector,

$$e(x) = e^{i\alpha(x)} \begin{pmatrix} x_1(x) \\ x_2(x) + iy_2(x) \\ x_3(x) + iy_3(x) \end{pmatrix}, \quad (8)$$

where $x_1(x) \geq 0$. For $U(2)$, one may use two orthonormal complex five-vectors[2].

We have successfully used this method to simulate $U(1)$ and are now applying it to $SU(2)$.

5. FERMIONS WITHOUT DOUBLERS

On the lattice each species of fermion appears as 16 different fermions. The root of this problem is that the natural discretization of the Fermi action approximates the derivative by means of a gap of two lattice spacings.

At the price of some nonlocality, we may leave out the unwanted states from the start[3]. Thus on a lattice of even size $N = 2F$, we may place independent fermionic variables $\psi(2n)$ and $\bar{\psi}(2n)$ only on the F^4 even sites $2n$ where n is a four-vector of integers,

$$2n = (2n_1, 2n_2, 2n_3, 2n_4) \quad (9)$$

and $1 \leq n_i \leq F$ for $i = 1, \dots, 4$. To extend the variables $\psi(2n)$ and $\bar{\psi}(2n)$ to the nearest-neighbor sites $2n \pm \hat{\mu}$, we first define the Fourier variables $\tilde{\psi}(k)$ (and $\tilde{\bar{\psi}}(k)$),

$$\tilde{\psi}(k) = \frac{1}{F^2} \sum_n \exp \left[-i \frac{2\pi n \cdot k}{F} \right] \psi(2n) \quad (10)$$

in which the sum extends over each $n_i, i = 1, \dots, 4$ from 1 to F and k is a four-vector of integers $k = (k_1, k_2, k_3, k_4)$ with $1 \leq k_i \leq F$. In terms of these Fourier variables, the dependent, nearest-neighbor variable $\psi(2n + \hat{\mu})$ is

$$\psi(2n + \hat{\mu}) = \frac{1}{F^2} \sum_k e^{i\pi(2n + \hat{\mu}) \cdot k / F} \tilde{\psi}(k). \quad (11)$$

in which the sum extends over each $k_i, i = 1, \dots, 4$ from 1 to F . In terms of the lattice delta functions

$$\delta^\pm(2j) = \frac{1}{F} \sum_{l=1}^F \exp \left[i \frac{2\pi (\pm \frac{1}{2} - j) l}{F} \right], \quad (12)$$

we may write the nearest-neighbor variables as the sums

$$\psi(2n \pm \hat{\mu}) = \sum_{j=1-n_\mu}^{F-n_\mu} \delta^\pm(2j) \psi(2n + 2j\hat{\mu}). \quad (13)$$

The action with independent fields at only F^4 sites is

$$S = (2a)^4 \sum_n \bar{\psi}(2n) \{ -m \psi(2n) - \sum_{\mu=1}^4 \frac{\gamma_\mu}{2ai} [\psi(2n + \hat{\mu}) - \psi(2n - \hat{\mu})] \}, \quad (14)$$

in which the nearest-neighbor variables $\psi(2n \pm \hat{\mu})$ are given by eq.(13) and the sum is over $1 \leq n_i \leq F$ as in eq.(10).

We may now verify that there are no doublers. The Fourier series (10) and (11) diagonalize the action (14) and the lattice propagator is

$$\frac{-ma + \sum_\mu \gamma_\mu \sin \left(\frac{\pi k_\mu}{F} \right)}{m^2 a^2 + \sum_\mu \sin^2 \left(\frac{\pi k_\mu}{F} \right)}. \quad (15)$$

For $m = 0$ this propagator has poles only at $k_\mu = F$, which is the same point as $k_\mu = 0$. There are

no doublers. Gauge fields may be added in the usual way or in the vectorial manner of section 4. In a gauge theory with a gauge field $U_\mu(n)$ defined on the link $(n, n + \hat{\mu})$, one may construct the ordered product $U(2n, 2n + 2j\hat{\mu})$ of Wilson links $U_\mu(n)$ from site $2n + 2j\hat{\mu}$ to site $2n$ for $j > 0$. Thus by inserting the line $U(2n, 2n + 2j\hat{\mu})$ into the action (14), one may render it covariant. This procedure should also work for chiral gauge theories.

Because of the lack of locality, the fermion propagator is not as sparse as the usual propagator. On the other hand, there are only one-sixteenth as many fermionic variables, and so the fermion propagator is smaller by a factor of 256. The present formalism of thinned fermions therefore may be useful in practice as well as in principle.

We intend to test this idea by simulating QED in two dimensions.

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REFERENCES

- [1] M. Creutz, *Phys. Rev. D* 21 (1980) 2308; *Phys. Rev. Letters* 45 (1980) 313.
- [2] M. Dubois-Violette, Y. Georgelin, *Phys. Lett.* 82B (1979) 251; K. Cahill and S. Raghavan, *J. Phys. A* 26 (1993) 7213.
- [3] K. Cahill, hep-lat/950813; see also S. Zenkin, *Phys. Lett.* B366 (1996) 261.

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